Support and Associated primes

Algebra Motivation:

Let n be an integer.

We can write its unique prime factorization as

$$n = \pm p_1^{d_1} \cdots p_t^{d_t}.$$

In fact, in $\mathbb{Z}$, $(n) = (p_1^{d_1}) \cap \dots \cap (p_t^{d_t})$.

We will see that the "associated primes" of (n) are the (p_i) and the primary components of (n) are the $(p_i^{d_i})$.

We will use these concepts to generalize the unique factorization of integers to arbitrary rings.

Geometric motivation:

Let $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ an ideal.

Def: The closed set $V(I)$ is reducible if it can be written $V(I) = V(I') \cup V(I'')$ where $V(I)$ is not equal to $V(I')$ or $V(I'')$.

Otherwise $V(I)$ is irreducible.

Claim: $V(I)$ is irreducible $\Leftrightarrow$ $\sqrt{I}$ is prime.

Pf: If $\sqrt{I}$ is prime, suppose

$$V(I) = V(\sqrt{I}) = V(J_1) \cup V(J_2)$$

Then WLOG, $\sqrt{I} \in V(J_1)$, so $V(\sqrt{I}) \subseteq V(J_1)$

$$\Rightarrow V(I) = V(J_1).$$

So $V(I)$ is irreducible.

If $\sqrt{I}$ is not prime, take $f, g \notin \sqrt{I}$ s.t. $fg \in \sqrt{I}$.

Then if $P \in V(\sqrt{I})$, $fg \in P \Rightarrow f \in P$ or $g \in P$.

So $V(I) = V(f, \sqrt{I}) \cup V(g, \sqrt{I})$, neither of which is equal to $V(I)$, since $\sqrt{I} = \bigcap_{\substack{P \supset I \\ \text{prime}}} P \not\ni f, g$. $\square$

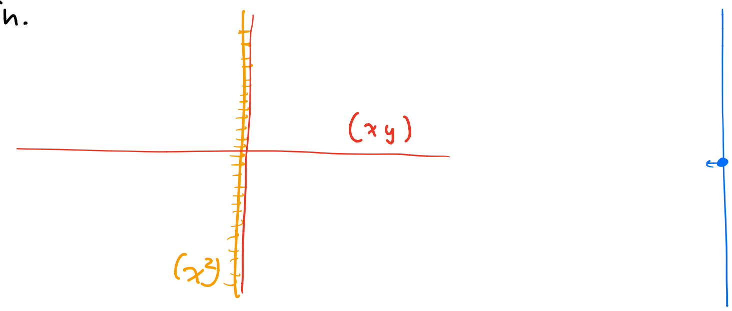
If I is radical, we will see that it can actually be written as a finite intersection of prime ideals in a unique minimal way. This is the "primary decomposition" of I and is equivalent to writing $V(I)$ in the unique minimal way of the union of closed sets.

The situation is more complicated if I is not radical:

Ex: Define $I := (x^2, xy) \subseteq k[x,y]$.

Notice that $I = (x) \cap (x^2, y)$.

Geometrically, the "scheme" $V(I)$ is (roughly) the line $x=0$ w/ additional structure (i.e. a tangent direction) at the origin.



We will see, purely algebraically, that this is reflected in the associated primes, which are (x) and (x,y).

However, we can write $\underbrace{I = (x) \cap (x^2, y)}_{\text{radical} = (x,y)}$ or $\underbrace{I = (x) \cap (x^2, xy, y^2)}_{\text{radical} = (x,y)}$

So the description as the intersection of ideals whose radicals are the associated primes is not unique.

Support of a module

Let M be an R-module.

Def: The support of M is the subset

$$\operatorname{Supp} M := \{ p \in \operatorname{Spec} R \mid M_p \neq 0 \} \subseteq \operatorname{Spec} R.$$

Recall that we showed $M = 0 \iff M_m = 0$ $\forall$ max'l $m \subseteq R$.

However $M = 0 \Rightarrow M_p = 0$ $\forall$ primes $P \subseteq R$, which implies $M_m = 0$ $\forall$ max'l $m \subseteq R$, so one can replace "maximal" with "prime." That is,

$$\operatorname{Supp} M = \emptyset \iff M = 0.$$

Also notice that if $P \in \operatorname{Supp} M$ and $Q \in V(P)$, then the following commutes:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_P \\ M_Q \arrow[ur] & \end{tikzcd}$$

and $M_P \neq 0 \Rightarrow M \neq 0$, so $M_Q \neq 0$ and thus $V(P) \subseteq \operatorname{Supp}(M)$.

Ex: If $I \subseteq R$ is an ideal, let $M = R/I$.

What is $\operatorname{Supp} M$?

If $P \subseteq R$ is prime, then

$$M_P \cong R_P / IR_P,$$

which is $0 \iff IR_P$ contains a unit $\iff I \not\subseteq P$.

Thus, $\operatorname{Supp}(R/I) = V(I) \subseteq \operatorname{Spec} R$.

More generally, we can give the following description of $\operatorname{Supp} M$:

Prop: If M is finitely generated, then

$$\operatorname{Supp} M = V(\operatorname{Ann} M) \subseteq \operatorname{Spec} R.$$

Pf: Suppose M is generated by $m_1, \dots, m_n$.

Then $r \in \operatorname{Ann}(M) \iff r \in \operatorname{Ann}(m_i)$ for each i.

Thus $\operatorname{Ann}(M) = \bigcap_{i=1}^{n} \operatorname{Ann}(m_i)$.

$$P \notin \operatorname{Supp}(M) \iff \text{for each } i,\ \exists\, u_i \notin P \text{ s.t. } u_i m_i = 0$$
$$\iff \operatorname{Ann}(m_i) \not\subseteq P \quad \forall\, m_i$$

So $P \in \operatorname{Supp}(M) \iff \operatorname{Ann}(m_i) \subseteq P$ for some i.

$$\iff P \in \bigcup_i V(\operatorname{Ann}(m_i)) = V\left(\bigcap \operatorname{Ann}(m_i)\right)$$
$$= V(\operatorname{Ann}(M)). \quad \square$$

Note that finite generation is necessary:

Ex: Let $M = \bigoplus_{\substack{p \neq 0 \\ \text{prime}}} \mathbb{Z}/(p)$. Then for $Q \in \operatorname{Spec} \mathbb{Z}$

$$M_p = \bigoplus (\mathbb{Z}/(p))_Q$$

So $\operatorname{Supp} M = \bigcup_p \operatorname{supp}\left(\mathbb{Z}/(p)\right) = \bigcup \{(p)\} = \operatorname{Spec} \mathbb{Z} \setminus \{(0)\}$,

which is not closed.

Associated Primes

Def: A prime P of R is associated to M if there is some $x \in M$ s.t. $P = \operatorname{Ann}(x) = \{r \in R \mid rx = 0\}$.

The set of all primes associated to M is denoted $\operatorname{Ass}_R M$, or just $\operatorname{Ass} M$ if the ring is clear.

Caution: Sometimes the associated primes of R/I over R are just called the associated primes of I.

Remark: If $P \in \operatorname{Ass} M$, then $P = \operatorname{Ann}(x)$, so

$R \xrightarrow{\cdot x} M$ has kernel P, so $R/P \cong$ a submodule of M.

Conversely, if P is some prime ideal s.t. $R/P \hookrightarrow M$ as modules, then P is the annihilator of the image of 1.

That is:

P is an associated prime of M $\iff$ R/P is isomorphic to a submodule of M.

Ex: If R is an integral domain, then $\operatorname{Ass}_R R = \{(0)\}$.

Claim: For any R-module M, we have

$$\mathrm{Ass}\, M \subseteq \mathrm{Supp}\, M.$$

Pf: Suppose $P \in \mathrm{Ass}\, M$. Then we have an injection

$$R/P \hookrightarrow M$$

Localizing preserves injections, so

$$\underbrace{(R/P)_P}_{\neq 0} \hookrightarrow M_P \implies M_P \neq 0 \implies P \in \mathrm{Supp}\, M. \square$$

Ex: Let $R = \mathbb{C}[x,y]$ and $M = \mathbb{C}[x,y]/(x^2, xy)$.

$$\mathrm{Supp}\, M = V(x^2, xy) = V(x) = \{(x)\} \cup \{(x, y-a)\}.$$

Which of these is in $\mathrm{Ass}\, M$?

$$f \in \mathrm{Ann}(x) \iff fx = ax^2 + bxy = x(ax + by)$$

$$\iff f = ax + by \quad (\text{since } \mathbb{C}[x,y] \text{ is an integral domain})$$

$$\iff f \in (x, y).$$

Similarly, $f \in \mathrm{Ann}(y) \implies fy = ax^2 + bxy$

$$\implies ax^2 = fy - bxy = y(f - bx)$$

Since $\mathbb{C}[x,y]$ is a UFD, $a = a'y$.

$$\text{So} \quad fy = y(a'x^2 + bx)$$

$$\Rightarrow f \in (x).$$

Conversely, $x \in \mathrm{Ann}(y)$, so $\mathrm{Ann}(y) = (x)$.

We will see that these are the only two associated primes.

Now we state some important results about associated primes

Theorem: Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then

a.) $\mathrm{Ass}\, M$ is finite and nonempty, each containing $\mathrm{Ann}(M)$. It includes all primes minimal among those containing $\mathrm{Ann}\, M$.

b.) $$\bigcup_{P \in \mathrm{Ass} M} P = \underbrace{\{\text{zerodivisors on } M\}}_{\substack{\| \\ r \in R \text{ s.t. } rm = 0 \\ \text{for some } m \neq 0 \text{ in } M.}} \cup \{0\}$$

We'll prove this in the next section after a few lemmas.

Remark: Why can we find primes minimal over an ideal?

Let $\{Q_i\}$ be a chain of prime ideals containing I.

Then if $ab \in \bigcap Q_i$, a or b is in all Q_i, so $\bigcap Q_i$ is prime.

That is, every chain has a lower bound, so Zorn's Lemma implies that there exist minimal primes over I.

(Note that this holds for even non-Noetherian rings!).

Def: The primes in $\operatorname{Ass} M$ that are not minimal are called <u>embedded</u> primes of M.

If $M = R/I$, then if P is an embedded prime in R, $V(P)$ is called an <u>embedded component</u> of $\operatorname{Spec}(R/I)$.

If P is a minimal associated prime, $V(P)$ is an <u>isolated component</u> of $\operatorname{Spec}(R/I)$.

Ex: In the $I = (xy, x^2)$ example, (x) is an isolated component and (x, y) an embedded component.